Dynamic Response of Rapidly Heated Plate Elements

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This study concerns analysis of displacement and stress histories for the dynamic thermoelastic response of rapidly heated plate elements. A variational principle is developed in which the total state of strain and the state of bending stress are varied simultaneously to seek a stationary value of a dynamic thermoelastic functional. The midsurface stresses are not variationally determined since, by hypothesis, they are related to the midsurface strains through the constitutive law. Indirect application of the variational principle yields the classical equations of motion, the force-displacement boundary conditions, and the constitutive relationships between the distortions and bending (and twisting) stresses. A semidirect application of the variational principle that eliminates spatial-coordinate dependence yields generalized time-dependent ordinary differential equations of dynamic equilibrium and constitutive relations between the generalized force and displacement parameters of assumed spatial distributions. Results obtained using the dynamic thermoelastic variational principle demonstrate displacement and bending-moment-convergence characteristics far superior to conventional solutions. A study of the influence of temperature-dependent material properties shows that neglect of temperature dependence is an unconservative assumption; further it is demonstrated that incomplete consideration of temperature dependence can lead to dangerously unconservative results.

Nomenclature

$\bar{C}$ = thermoelastic coupling $= \frac{1}{1-\nu^2}\int_{-h/2}^{h/2} Ezdz$
$\bar{D}$ = flexural stiffness $= \frac{1}{1-\nu^2}\int_{-h/2}^{h/2} Ez^2dz$
E = Young's modulus of elasticity
F' = complementary-energy density
$\bar{H}$ = extensional stiffness $= \frac{1}{1-\nu^2}\int_{-h/2}^{h/2} Edz$
M_T = thermal moment $= -\frac{1}{1-\nu}\int_{-h/2}^{h/2} E\alpha\theta zdz$
M_x,M_y,M_{xy} = stress couples
M_{xB},M_{yB},M_{xyB} = bending stress couples
M_{xK},M_{yK},M_{xy} = kinematic stress couples
M_{xS},M_{yS},M_{xyS} = stretching stress couples
M_{x22},M_{x24},M_{x42}, M_{y22},M_{y24},M_{y42}, $M_{xy11},M_{xy13},M_{xy31}$ = coefficients of assumed bending- and twisting-moment functions
N_T = thermal thrust $= -\frac{1}{1-\nu}\int_{-h/2}^{h/2} E\alpha\theta dz$
N_x,N_y,N_{xy} = stress resultants
N_{xB},N_{yB},N_{xyB} = bending stress resultants
N_{xK},N_{yK},N_{xy} = kinematic stress resultants
N_{xS},N_{yS},N_{xyS} = stretching stress resultants
T = kinetic energy
U'' = Reissner functional
V = potential of applied loads
a = plate dimension in the x direction
b = plate dimension in the y direction
c = specific heat
h = plate thickness
k = thermal conductivity
t = time
t_1,t_2 = arbitrary values of time
u,v,w = middle surface displacements in the x, y, and z directions, respectively
u_1,v_1, w_{22},w_{24},w_{42} = coefficients of assumed displacement functions
x,y,z = geometric coordinates (see Fig. 1)
θ = temperature
α = coefficient of thermal expansion
γ_{xy} = shear strain
σ_x,σ_y = direct stress components
σ_{xB},σ_{yB} = bending components of direct stresses
σ_{xK},σ_{yK} = kinematic components of direct stresses
σ_{xS},σ_{yS} = stretching components of direct stresses
σ_{xT},σ_{yT} = thermal stress components $= -E\alpha\theta/(1-\nu)$
ϵ_x,ϵ_y = direct strain components
η = dimensionless geometric coordinate parameter $= 2y/b$
κ = thermal diffusivity
ν = Poisson's ratio
ξ = dimensionless geometric coordinate parameter $= 2x/a$
ρ = mass density
τ_{xy} = kinematic shear stress
τ_{xyB} = shear stress due to deformation out of middle surface
τ_{xyS} = shear stress due to deformation in middle surface

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Introduction

DYNAMIC response in thermoelastic systems was considered as early as 1837 when Duhamel[1] concluded that the time rate of change of temperature in practical systems is so slow that inertia terms can be disregarded in the equations of thermoelasticity. This position represented the state of the art for over 100 years. In the middle of this century, advances in supersonic flight, rocket propulsion, re-entry technology, and nuclear science prompted the scientific community to review its capability regarding dynamic thermoelasticity. The first analysis of dynamic response to rapid heating was performed in 1950 by Danilovskaya[2] who examined the state of stress in a half-space as a result of sudden heating of its surface. A significant and often referenced effort was carried out at Columbia University in 1956 and 1957 by Boley and Barber[3,4] who studied thermally induced vibrations of beams and plates. Boley and Barber found that inertia effects are important when the fundamental period of vibration is, at least, of the same order of magnitude as the thermal period h^2/κ. More recent investigations of thermally induced vibrations in other structural configurations by Kraus[5] (spherical shells) and by Lu and Sun[6] (conical shells) have supported, qualitatively, the findings of Boley and Barber. In 1968, McQuillen[7] examined the thermally excited dynamic response of beams and cylindrical shells with the use of the fully coupled equations of dynamic thermoelasticity, including the heating effect of mechanical energy dissipation; and he concluded that the fully coupled theory was required only in those instances where a resonance condition was induced by a moving thermal input.

The subject of dynamic response of rapidly heated structures has at least two areas in which important information has been lacking. First, while existing theory and analysis are adequate for dealing with displacement phenomena, they are not efficient in predicting stress behavior. This is recognized as a serious shortcoming when it is considered that a majority of structural design decisions are based on stress rather than displacement considerations. Second, it is well known that certain material properties (in particular, Young's modulus and the coefficient of thermal expansion) change significantly in a high-temperature environment. In spite of the important influence this temperature dependence must have on dynamic response characteristics, little formal treatment of the subject has been presented.

The variational principle originated by Reissner,[8] wherein the states of both stress and strain are established simultaneously, offers a means of analyzing stress distributions in deflection, stability, and vibration problems. The present study uses a dynamic thermoelastic variational principle, which is a modification of the Reissner principle, to determine both stresses and deformations of rapidly heated plates. Temperature-dependent physical properties are inherent in the analysis; they are manifested in the variable coefficients of the variationally derived equations of motion.

This study examines the linear dynamic thermoelastic response of a rectangular plate, with temperature-dependent material properties, subjected to an arbitrary temperature distribution history. Displacement and moment histories are computed for a square, simply supported plate that is instantaneously subjected to a uniform heat flux acting across one surface, while the other surface is assumed to be perfectly insulated.

Theoretical Considerations

Basic Assumptions

It is assumed that the so-called semicoupled theory of dynamic thermoelasticity is adequate for this study, and that the temperature distribution history is a known quantity.

Strain-Displacement Relations

The strain-displacement relations for a rectangular plate (see Fig. 1) are

$$\begin{aligned}\epsilon_x &= \partial u/\partial x - z\partial^2 w/\partial x^2\\ \epsilon_y &= \partial v/\partial y - z\partial^2 w/\partial y^2\\ \gamma_{xy} &= \partial u/\partial y + \partial v/\partial x - 2z\partial^2 w/\partial x\partial y\end{aligned}\tag{1}$$

Constitutive Law

The stress-strain relations are the thermal Hooke's law.

$$\begin{aligned}\sigma_x &= E(\epsilon_x + \nu\epsilon_y)/(1-\nu^2) - E\alpha\theta/(1-\nu)\\ \sigma_y &= E(\epsilon_y + \nu\epsilon_x)/(1-\nu^2) - E\alpha\theta/(1-\nu)\\ \tau_{xy} &= E\gamma_{xy}/2(1+\nu)\end{aligned}\tag{2}$$

Young's modulus and the coefficient of thermal expansion are treated as arbitrary functions of temperature, but Poisson's ratio is assumed to be independent of the thermal environment.

Dynamic Thermoelastic Variational Principle

The basis of this analysis is Hamilton's principle extended to deformable bodies with the strain energy replaced by the Reissner functional. In symbolic form, the principle is given by

$$\delta\int_{t_1}^{t_2}(T - U'' - V)dt = 0 \tag{3}$$

where the Reissner functional is

$$U'' = \int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\int_{-h/2}^{h/2}(\sigma_x\epsilon_x + \sigma_y\epsilon_y + \tau_{xy}\gamma_{xy} - F')dzdydx \tag{4}$$

If the applicability of the thermal Hooke's law is presumed, then the complementary-energy density can be written as

$$F' = (1/2E)[\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y + 2(1+\nu)\tau_{xy}^2 + 2E\alpha\theta(\sigma_x + \sigma_y)] \tag{5}$$

and the variational principle in the absence of external forces can be written as

$$\delta\int_{t_1}^{t_2}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\int_{-h/2}^{h/2}\left\{\frac{\rho}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] - \sigma_x\epsilon_x - \sigma_y\epsilon_y - \tau_{xy}\gamma_{xy} + \frac{1}{2E}[\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y + 2(1+\nu)\tau_{xy}^2 + 2E\alpha\theta(\sigma_x+\sigma_y)]\right\}dzdydxdt = 0 \tag{6}$$

It is advantageous now to divide the direct stress field into two components; the thermal stress, which is defined as

$$\sigma_{xT} = \sigma_{yT} = -E\alpha\theta/(1-\nu) \tag{7}$$

and the kinematic stress (or that developing as a result of deformation), which is designated by the subscript K. Thus,

$$\sigma_x = \sigma_{xK} - E\alpha\theta/(1-\nu) \text{ and } \sigma_y = \sigma_{yK} - E\alpha\theta/(1-\nu) \tag{8}$$

The problem can be reduced to one of two dimensions when Eqs. (8), along with the strain-displacement relations of Eq. (1), are substituted into Eq. (6), and integration through the thickness of the plate is carried out. The statement of the variational principle becomes

$$\begin{aligned}\delta\int_{t_1}^{t_2}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}&\left\{\frac{\rho h}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] - (N_{xK}+N_T)\frac{\partial u}{\partial x} + (M_{xK}+M_T)\frac{\partial^2 w}{\partial x^2} - (N_{yK}+N_T)\frac{\partial v}{\partial y} + \right.\\ &(M_{yK}+M_T)\frac{\partial^2 w}{\partial y^2} - N_{xy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + 2M_{xy}\frac{\partial^2 w}{\partial x\partial y} - \\ &\frac{1}{1-\nu^2}\frac{1}{1-\bar{C}^2/\bar{H}\bar{D}}\left(\frac{1}{\bar{H}}\left[\frac{N_{xK}^2}{2} + \frac{N_{yK}^2}{2} - \nu N_{xK}N_{yK} + (1+\nu)N_{xy}^2\right] + \frac{1}{\bar{D}}\left[\frac{M_{xK}^2}{2} + \frac{M_{yK}^2}{2} - \nu M_{xK}M_{yK} + \right.\right.\\ &\left.(1+\nu)M_{xy}^2\right] - \frac{\bar{C}}{\bar{H}\bar{D}}\left[N_{xK}M_{xK} + N_{yK}M_{yK} - \nu(N_{xK}M_{yK} + N_{yK}M_{xK}) + 2(1+\nu)N_{xy}M_{xy}\right]\Bigg) - \\ &\left.\frac{1}{1-\nu}\int_{-h/2}^{h/2}E\alpha^2\theta^2 dz\right\}dydxdt = 0\end{aligned}\tag{9}$$

The requirement that the first variation of the integral expression in Eq. (9) vanishes with respect to each of the displacement parameters gives the equations of motion and attendant boundary conditions. Simultaneously, equating to zero the first variations with respect to the stress resultants and couples yields constitutive relations between the stress resultants and couples and the middle-surface strains and curvatures (and twist).

Some significant advances in structural analysis have been made by exploiting Reissner's principle in its original form and various modifications thereof.[9–14] One of the basic precepts underlying the modifications is that in problems in which the distribution of bending stresses is either easily determined or of little consequence in effecting satisfaction of middle-surface equilibrium, the bending stresses can be related to the curvatures directly according to the constitutive law. Then, with the bending stresses eliminated as unknown quantities, the solution for the usually complex distribution of inplane stresses is expedited. In the present application, a reversed situation occurs. Although the nature of the inplane stress distribution is relatively explicit, the behavior of the bending stresses is very complex. Therefore, this problem of dynamic thermoelastic response may be simplified by determining the middle-surface stresses from the middle-surface strains [Eqs. (1) with $z = 0$] and the stress-strain law of Eq. (2).

With this simplification, the modified variational statement of Eq. (9) is obtained as

$$\delta \int_{t_1}^{t_2} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \left\{ \frac{\rho h}{2} \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2 \right] - \left[\frac{\bar{H}}{2} \left(\frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y} \right) + N_T \right] \frac{\partial u}{\partial x} + \left[\bar{C} \left(\frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y} \right) + M_{xB} + M_T \right] \frac{\partial^2 w}{\partial x^2} - \left[\frac{\bar{H}}{2} \left(\frac{\partial v}{\partial y} + \nu \frac{\partial u}{\partial x} \right) + N_T \right] \frac{\partial v}{\partial y} + \left[\bar{C} \left(\frac{\partial v}{\partial y} + \nu \frac{\partial u}{\partial x} \right) + M_{yB} + M_T \right] \frac{\partial^2 w}{\partial y^2} - \frac{\bar{H}}{2} \frac{1-\nu}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \left[\bar{C}(1-\nu) \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + 2M_{xyB} \right] \frac{\partial^2 w}{\partial x \partial y} + \frac{1}{1-\nu^2} \frac{1}{2\bar{D}} [M_{xB}^2 + M_{yB}^2 - 2\nu M_{xB} M_{yB} + 2(1+\nu) M_{xyB}^2] - \frac{1}{1-\nu} \int_{-h/2}^{h/2} E\alpha^2\theta^2 dz \right\} dy dx dt = 0 \tag{10}$$

where the subscript B indicates the stress couples that result from bending of the middle surface. In addition to the thermal moment and the ordinary bending moment, the stress couple has, as a result of the coupling term $\bar{C}$, a component due to extension of the middle surface.

Method of Solution

Direct variational solutions are based on the concept that when a functional has been developed from which the governing equations of a specific problem can be derived by extremizing that functional with respect to the dependent variables, an approximate solution, which can be improved to any degree of accuracy, is possible. The procedure for obtaining the approximate solution is to substitute an assumed (spatially defined) solution into the functional and extremize the resulting spatially integrated functional with respect to the arbitrary parameters appearing in the assumed solution. The assumed solution must, for reasons of continuity, satisfy the variationally determined geometric boundary conditions.

In the direct variational solution of dynamic response problems, it is common to assume harmonic solutions for the time-dependent portions of the assumed functions. In the present analysis, the governing equations, as a result of the temperature-dependent nature of the material properties, have variable coefficients. This situation makes the assumption of harmonic oscillations unrealistic and precludes a direct variational approach. Instead, a semidirect method is employed in which the free parameters are treated as unknown functions of time. The resulting variation with respect to these unknown quantities yields a set of ordinary differential equations in time.

Two-Step Solution

A two-step procedure is used to compute the numerical results presented herein. First, the so-called thermoelastic solution is obtained by disregarding inertial effects; then, the dynamic portion of the solution is determined and combined with the thermoelastic solution to give the total response. This two-step solution is employed because the variational theorem developed earlier is particularly advantageous in solving the thermoelastic problem; whereas a semi-direct variational solution of a minimum-potential-energy formulation is best suited for the dynamic phase of the problem. The Reissner variational principle is advantageous when the deformation pattern must be represented by a series. Rather than differentiating this series term by term and applying the constitutive law to form another, certainly more slowly converging, series for the moment distribution (as in the potential-energy approach), it is extremely efficient to assume a separate moment series, independent of the displacement series, and determine all coefficients variationally. Such a situation occurs in the case of the thermoelastic solution. The dynamic displacement history is expressed also in series form; that is, a series of the orthogonal mode shapes each of which can be described by a single product of trigonometric functions. Every mode shape (or displacement eigenfunction) is inseparably linked, through a common function of time, with a corresponding moment eigenfunction. Since the mode shapes are each described by a single expression, their companion moment functions can be defined precisely by the moment-curvature relations; consequently, Reissner's approach provides no advantage in finding the dynamic solution. Were the modes to be so complex that they require a series expression for their description, Reissner's technique or modifications thereof would be very useful in developing independent series expressions for the moment eigenfunctions.

Temperature Distribution History

In each problem undertaken in this study, the thermal excitation is a suddenly applied uniform heat flux acting over one surface; whereas the other surface is presumed to be perfectly insulated. These conditions are exactly those analyzed by Boley and Barber,[3,4] by Kraus,[5] and by Lu and Sun.[6] They are selected here for convenient comparison of present results with those of previous analyses. For this application a heat flux of 10 Btu/(in.^{2}-sec) is selected.

Geometric and Material Properties

To establish confidence in the semidirect variational procedure, the temperature dependence of material properties is at first ignored, and the results are compared directly with those of Boley and Barber (Figs. 2 and 3).

The dimensions of the plate analyzed (10 in. × 10 in. × 0.06 in.) have been selected as representative of contemporary aerospace structural elements and the material specified (7075-T6 aluminum) in a reasonable choice for light weight, high-strength, high-temperature applications. Material

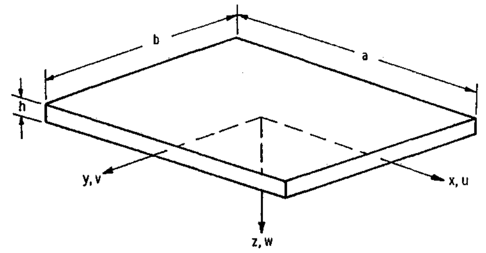

Fig. 1 Geometry and coordinate system.

properties of the aluminum have been obtained from Ref. 15 and are as follows: $E = 10.5 \times 10^6$ psi; $\alpha = 12.4 \times 10^{-6}$ (in./in.)/°F; $\nu = 0.33$; $\rho = 0.000261$ lb-sec²/in.⁴; $k = 0.00176$ Btu/(in.-sec-°F); $c = 88.9$ (Btu-in.)/(lb-sec²-°F); and $\kappa = 0.0758$ in.²/sec.

Thermoelastic Displacement Functions

The assumed displacement functions for the thermoelastic solution are

$$\begin{aligned} u &= u_1\xi \\ v &= v_1\eta \\ w &= w_{22}(1-\xi^2)(1-\eta^2) + w_{24}(1-\xi^2)(\eta^2-\eta^4) + \\ &\qquad w_{42}(\xi^2-\xi^4)(1-\eta^2) \end{aligned} \tag{11}$$

It is apparent that for the thermoelastic response of a plate with unrestrained edges, the solutions assumed for inplane displacement are exact. The transverse-displacement expression is basically a fourth-degree (in each spatial variable) polynominal with all except three of the coefficients eliminated by enforcing geometric boundary conditions and symmetry requirements.

Thermoelastic Bending Moment Functions

Although compatibility considerations dictate strict adherence to geometric boundary conditions in a semidirect variational solution, satisfaction of natural boundary conditions is not mandatory. However, in the interest of accelerated convergence and quality of the overall results, the assumed solution should attempt to satisfy as nearly as possible the natural boundary conditions. To this end, the distributions of bending and twisting moments for the thermoelastic solution are assumed as

$$\begin{aligned} M_{xB} &= -M_T + (\bar{C}/\bar{H})N_T + [(1-\nu)[M_T - \\ &\quad (\bar{C}/\bar{H})N_T] - M_{x22}](1-\xi^{10})\eta^2 + M_{x22}\times \\ &\quad [1-\xi^2(1-\eta^{10})] + M_{x24}(1-\xi^2)(\eta^2-\eta^4) + \\ &\quad M_{x42}(\xi^2-\xi^4)(1-\eta^2) \\ M_{yB} &= -M_T + (\bar{C}/\bar{H})N_T + \{(1-\nu)[M_T - \\ &\quad (\bar{C}/\bar{H})N_T] - M_{y22}\}\xi^2(1-\eta^{10}) + M_{y22}\times \\ &\quad [1-(1-\xi^{10})\eta^2] + M_{y24}(\xi^2-\xi^4)(1-\eta^2) + \\ &\quad M_{y42}(1-\xi^2)(\eta^2-\eta^4) \\ M_{xyB} &= M_{xy11}\sin(\pi/2)\xi\sin(\pi/2)\eta + \\ &\quad M_{xy13}\sin(\pi/2)\xi\sin(3\pi/2)\eta + \\ &\quad M_{xy31}\sin(3\pi/2)\xi\sin(\pi/2)\eta \end{aligned} \tag{12}$$

Salient considerations underlying the selection of these distributions are as follows.

Since the plate is simply supported, the total moment in the x direction must vanish at the edge $x = a/2$. So, by virtue of the classical moment-curvature relations,

$$M_T - \bar{D}[\partial^2w/\partial x^2 + \nu\partial^2w/\partial y^2]_{x=a/2} = 0 \tag{13}$$

But,

$$\partial^2w/\partial y^2|_{x=a/2} = 0 \tag{14}$$

Thus,

$$\bar{D}\,\frac{\partial^2w}{\partial x^2}\bigg|_{y=a/2} = M_T \tag{15}$$

When the moment in the y direction at the boundary $y = b/2$ is examined, it can be concluded similarly that

$$\bar{D}\,\frac{\partial^2w}{\partial y^2}\bigg|_{y=b/2} = M_T \tag{16}$$

Along this boundary the moment in the x direction is the sum of its thermal and kinematic components; that is,

$$M_x|_{y=b/2} = M_T - \bar{D}[\partial^2w/\partial x^2 + \nu\partial^2w/\partial y^2]_{y=b/2} \tag{17}$$

But,

$$\partial^2w/\partial y^2|_{y=b/2} = 0 \tag{18}$$

Thus,

$$M_x|_{y=b/2} = (1-\nu)M_T \tag{19}$$

In summary, the moment in the x direction vanishes along the edge $x = a/2$, but it has a constant finite value along the edge $y = b/2$. Consequently, the moment distribution must be discontinuous at the corner $x = a/2$, $y = b/2$ (see Fig. 4).

For an efficient semidirect variational solution, it is necessary to select a kinematic, thermoelastic moment function which closely simulates this complex distribution. For the present problem the kinematic, thermoelastic moment is assumed to be

$$\begin{aligned} M_{xK} &= -M_T + [(1-\nu)M_T - M_{x22}]\times \\ &\quad (1-\xi^m)\eta^2 + M_{x22}[1-\xi^2(1-\eta^n)] \end{aligned} \tag{20}$$

It can be verified that, with appropriate values of m and n, Eq. (20) provides an accurate simulation of the sought-after boundary values and gives a good approximation to the discontinuity at the corner. Selection of the most appropriate values of m and n can be accomplished variationally; that is, they may be treated as additional free parameters and determined through extremization of the dynamic thermoelastic functional. This procedure, however, introduces nonlinearity into an otherwise linear problem. For this reason it is more expedient to select m and n by a trial

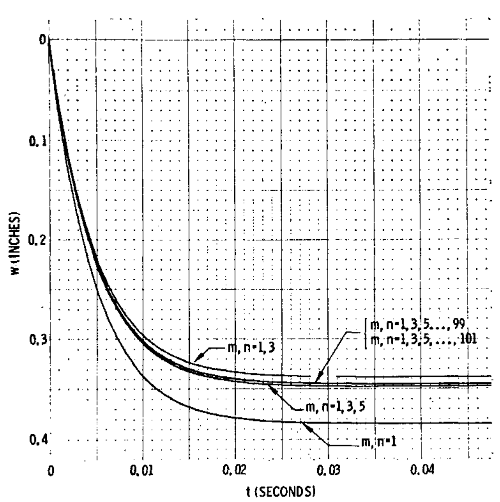

Fig. 2 Composite of Boley and Barber's histories of thermoelastic lateral displacement at the center of the plate.

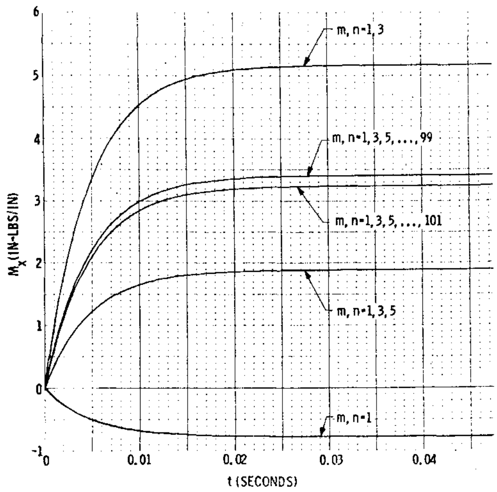

Fig. 3 Composite of Boley and Barber's histories of thermoelastic moment at the center of the plate.

and error process. As indicated in Eq. (12), a value of 10 was selected for both m and n.

For problems in which the edges of the plate are not restrained from inplane displacement, the moment due to stretching is given by

$$M_{xS} = -(\bar{C}/\bar{H})N_T \tag{21}$$

If the kinematic moment is separated into its stretching and bending components and the value of the moment at the boundary $y = b/2$ is adjusted accordingly, then the appropriate expression for the bending moment in the x direction is

$$M_{xB} = -\left(M_T - \frac{\bar{C}}{\bar{H}}N_T\right) + \left[(1-\nu)\left(M_T - \frac{\bar{C}}{\bar{H}}N_T\right) - M_{x22}\right](1-\xi^m)\eta^2 + M_{x22}[1 - \xi^2(1-\eta^n)] \tag{22}$$

To provide some flexibility for the moment distribution on the interior of the plate, without affecting boundary values, the above is augmented by two additional terms such that

$$M_{xB} = -[M_T - (\bar{C}/\bar{H})N_T] + \{(1-\nu)[M_T - (\bar{C}/\bar{H})N_T] - M_{x22}\}(1-\xi^m)\eta^2 + M_{x22}[1 - \xi^2(1-\eta^n)] + M_{x24}(1-\xi^2)(\eta^2-\eta^4) + M_{x42}(\xi^2-\xi^4)(1-\eta^2) \tag{23}$$

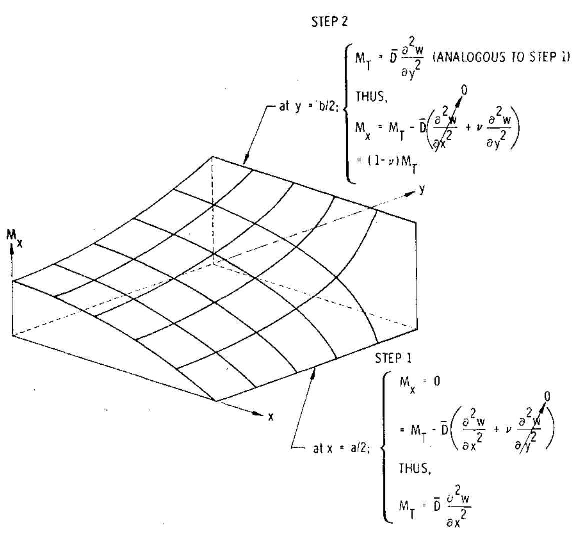

Fig. 4 Thermoelastic moment distribution.

An identical development gives the function assumed to represent the bending moment in the y direction.

The assumed twisting-moment distribution has been selected because, in addition to satisfying the condition of symmetry about the coordinate axes, it satisfies also the subtle natural boundary conditions concerning twisting moments. Along the edge $x = a/2$, the curvature in the x direction is, as shown earlier [Eq. (15)], a constant; that is,

$$\partial^2 w/\partial x^2|_{x=a/2} = M_T/\bar{D} \tag{24}$$

Thus,

$$\partial(\partial^2 w/\partial x^2)/\partial y_{x=a/2} = 0 \tag{25}$$

or, since the lateral displacement is a continuous function,

$$\partial(\partial^2 w/\partial y^2)/\partial x_{x=a/2} = 0 \tag{26}$$

Then, by virtue of the classical twisting moment-twist relation

$$\partial M_{xy}/\partial x|_{x=a/2} = 0 \tag{27}$$

It may be shown similarly that

$$\partial M_{xy}/\partial y|_{y=b/2} = 0 \tag{28}$$

The assumed twisting-moment distribution [Eq. (12)] clearly satisfies these requirements.

Dynamic Displacement Functions

The dynamic solution is accomplished by semidirect application of Hamilton's principle extended to deformable bodies. The assumed displacement functions are

$$\begin{aligned} u &= u_1\xi \\ v &= v_1\xi \\ w &= \sum_{m=1,3,5}\ \sum_{n=1,3,5} w_{mn} \cos\frac{m\pi}{2}\,\xi \cos\frac{n\pi}{2}\,\eta \end{aligned} \tag{29}$$

Solutions for Temperature-Independent Material Properties

The results of the semidirect variational analysis for the thermoelastic and total (thermoelastic plus dynamic) responses with temperature-dependence of material properties neglected are illustrated in Fig. 5.

Solution for Temperature-Dependent Material Properties

Plots of Young's modulus and the coefficient of thermal expansion vs temperature for 7075-T6 aluminum are presented in Ref. 15. Polynominals are fit to these graphs giving a mathematical description of the temperature dependence for these material properties. Graphs of E and α as functions of

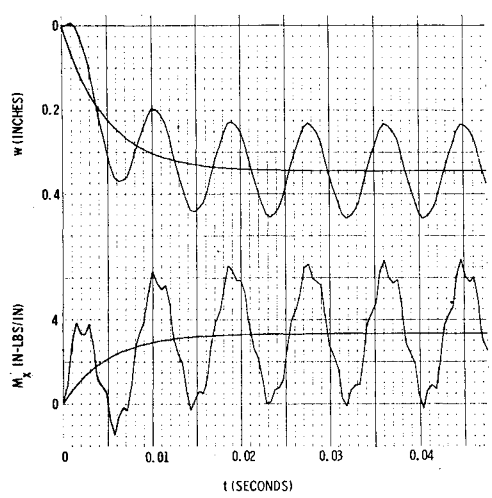

Fig. 5 Histories of thermoelastic and total lateral displacement and moment at the center of the plate; temperature-dependence effects neglected.

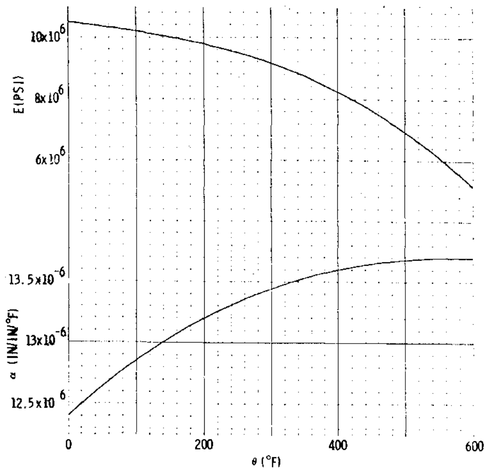

Fig. 6 Young's modulus and coefficient of thermal expansion vs temperature for 7075-T6 aluminum.

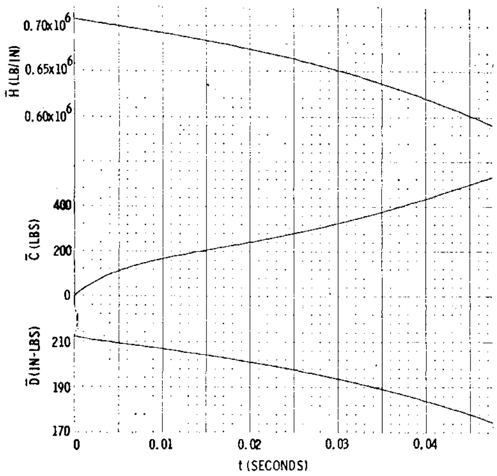

Fig. 7 Histories of stiffness characteristics.

temperature are given in Fig. 6. These quantities are used to compute histories of the stiffness characteristics $\bar{H}$, $\bar{C}$, and $\bar{D}$ (see Fig. 7). The thermal thrust N_T and thermal moment M_T are plotted vs time in Fig. 8.

The previous solution is repeated, except that now the coefficients of the resulting equations are time varying quantities. The results of this analysis are shown in Fig. 9.

Results and Discussion

The ability of previous analyses to satisfactorily predict displacement behavior and their inability to efficiently deal with stress histories is exemplified by the solution of Boley and Barber which has the form

$$w(x,y,t) = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} w_{mn}(t) \cos\frac{m\pi}{2}\,\xi \cos\frac{n\pi}{2}\,\eta \quad (30)$$

It has been observed from calculations that Boley and Barber's solution adequately describes the dynamic displacement and moment with a moderate number of terms from Eq. (30). Difficulties in achieving convergence for the total solution can be attributed wholly to the thermoelastic contribution. Inspection of the composite of thermoelastic displacement histories at the center of the plate shown in Fig. 2 indicates that the displacement expression converges rapidly. However, the composite histories of thermoelastic moment at the center of the plate shown in Fig. 3 reveal less satisfactory convergence behavior. It can be seen that the one-term solution for the thermoelastic moment is a negative quantity, though the correct value is positive. In fact, in proceeding from the 2500-term solution ($m,n = 1,3,5,\ldots, 99$) to the 2601-term solution ($m,n = 1,3,5,\ldots, 101$) the thermoelastic moment changes by about 6%.

Boley and Barber's equations were developed by equilibrium considerations and are expressed in terms of lateral displacement. The form of the series solution was selected to satisfy geometric boundary conditions. Under these circumstances, the analyst must accept for a kinematic moment solution whatever series expression evolves from the displacement quantity by application of bending (and twisting) moment-curvature (and twist) relations. In the case of Boley and Barber's plate problem, the resulting kinematic-moment solution is a series expression whose terms all vanish at the edges.

The kinematic thermoelastic moment must complement the constant (in the spatial sense) thermal moment to give the total thermoelastic moment as depicted in Fig. 4. Therefore, it is recognized that the kinematic moment is a function which has constant but different finite values along adjacent boundaries and exhibits a discontinuity at the plate corners. Describing this complex function with a series whose terms all vanish at the boundary is a prohibitive task.

Although they make no specific mention of the discontinuous corner moment, earlier researchers have recognized the difficulty of describing the finite kinematic boundary moment with a series whose terms vanish at the edges. Boley and Weiner[16] point out, "the results thus derived are not practical for computations pertaining to points near the edges of the plate." They suggest a solution of the type originally proposed by, among others, Nadai[17] (that is, replacement of the double trigonometric series with a single trigonometric series whose coefficients are arbitrary functions of space) to determine the moment in the vicinity of the boundary. However, it should be pointed out that the poor moment convergence discussed earlier is not near the boundary but at the center of the plate and probably represents the best solution anywhere in the plate.

A great advantage in applying the dynamic thermoelastic variational principle developed here is the opportunity to make use of what is known about the complex distribution of moments. The power of this approach is demonstrated by the resulting histories of lateral displacement and moment at the center of the plate illustrated in Fig. 5. The displacement solution is indistinguishable from the "converged" displacement solution of Boley and Barber. Furthermore, the moment history at the center of the plate computed by the variational technique is bounded by the 2500-term and the 2601-term solutions of Boley and Barber. This has

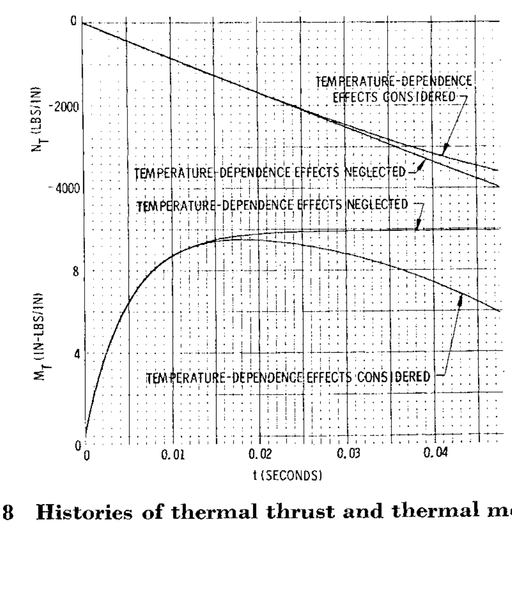

Fig. 8 Histories of thermal thrust and thermal moment.

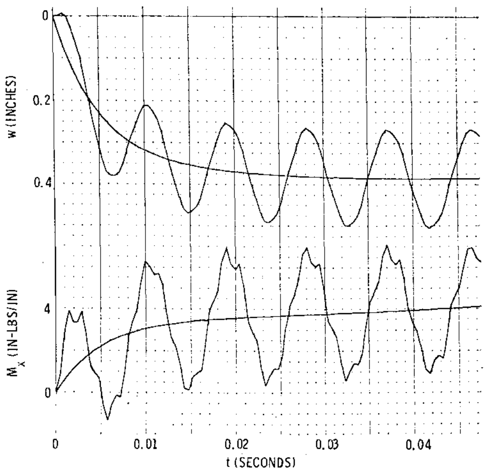

Fig. 9 Histories of thermoelastic and total lateral displacement and moment at the center of the plate.

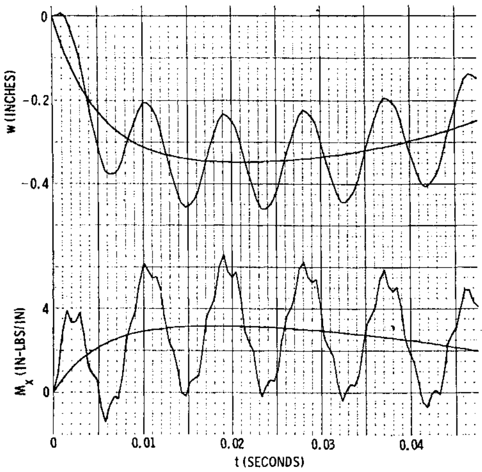

Fig. 10 Histories of thermoelastic and total lateral displacement and moment at the center of the plate; thermoelastic coupling neglected.

been accomplished by using assumed displacement and moment functions having a total of only 14-free parameters. In fact, if the inplane displacement parameters, which are irrelevent to this solution, are neglected and cognizance is taken of symmetry, then the variational solution requires only 7-independent parameters to produce an essentially converged solution; this solution is more accurate than that given by the classical solution after the consideration of over 2600 terms.

Results of the analysis, considering temperature-dependent physical properties, are shown in Fig. 9. Comparison of Figs. 5 and 9 indicates the temperature-dependence effects cause an increase in lateral displacement of about 10% at the end of one thermal period; that is, at $t \simeq 0.0475$ sec. The thermoelastic moment is, at the same time, increased by almost 25%. For this application, it is clear that neglecting the effect of temperature-dependent material properties is, from the standpoint of both displacement and moment prediction, both an inaccurate and unconservative process.

The following example demonstrates the danger of an incomplete treatment of temperature dependence. An analyst could recognize the importance of E and α varying with temperature from the standpoint of their influence on structural stiffness and thermal forces without sensing the significance of the thermoelastic coupling term $\bar{C}$. Then, his analysis would be carried out accounting for the variable nature of the inplane and bending stiffnesses and including the changes in the thermal thrust and moment, but, neglecting the interaction between inplane and lateral displacement caused by the thermoelastic coupling phenomenon. The results of such an analysis are presented in Fig. 10. Comparison of Figs. 9 and 10 shows that at the end of one thermal period neglect of the coupling term produces a thermoelastic displacement that is only about $\frac{3}{4}$ of the actual value. These fixures also show the incomplete analysis gives a thermoelastic moment which is only about $\frac{1}{2}$ the actual value after one thermal period. For this application, the neglect of the thermoelastic coupling is particularly unconservative, even more unconservative than disregarding temperature dependence altogether.

The significant differences between Figs. 9 and 10 are attributed to the thermoelastic coupling which permits the thermal thrust to influence the lateral displacement and, inherently, the bending moment. After one thermal period has elapsed, the quantity $-(\bar{C}/\bar{H})N_T$ is greater than the thermal moment, and the thermal thrust has become the dominant factor in the amplitude of lateral displacement. In Ref. 5, Kraus examined the dynamic response of rapidly heated spherical shells and compared his results qualitatively to those of Boley and Barber. He stated, "The difference in behavior between the shells and the beam and the plate is undoubtedly caused by the fact that only the shell analysis includes the coupling between in-plane and transverse effects and the attendant effect of the in-plane thermal force N_T." This statement is accurate for analyses which disregard the temperature-dependence effects; however, a thorough analysis of the beam and the plate also includes, as demonstrated herein, a coupling effect which introduces the influence of thermal thrust on bending behavior.

Concluding Remarks

It has been observed that preoccupation with displacement behavior and a priori satisfaction of Hooke's law in the development of previously existing theory has produced analytic procedures which are frequently inefficient. Results show that the variational approach developed here is more effective than conventional analyses for determining the response of rapidly heated plates; it is especially effective in predicting stress behavior. Finally, this investigation has established that neglect of the effects of temperature-dependent physical properties is an unconservative process and that neglect of thermoelastic coupling is particularly unconservative.

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